10 Information Theoretic Quantities

Tuesday, September 10, 2013
3:02 PM

Information Theoretic Quantities

- H (entropy)
- I (mutual info.)
- C (capacity)

Recall

convention : $0 \log 0 \equiv 0$

Entropy $H(X) = -\sum_{x} p_X(x) \log_2 p_X(x) = \mathbb{E}\left[-\log_2 p_X(X)\right]$

$\downarrow$

quantify/measure
- how random a RV is
- the amount of uncertainty a RV has
- how many bits (on average) to describe a RV

Fact: ① $H(X) \geq 0$

② $H(X)$ is maximized when $X \sim$ uniform

Ex. Uniform RV X on $1, 2, \ldots, n$

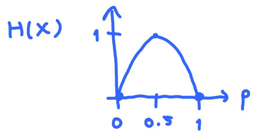
$$p_X(x) = \begin{cases} 1/n, & x = 1, 2, \ldots, n, \\ 0, & \text{otherwise}. \end{cases}$$

$$H(X) = -\sum_{x=1}^{n} \frac{1}{n} \log_2 \frac{1}{n} = \log_2 n$$

Ex. $X \sim \text{Bernoulli}(p)$

$$p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$H(X) = -(1-p)\log_2(1-p) - p\log_2 p \equiv H(p)$$



Ex.

$$X = \begin{cases} a & \text{with probability } \cancel{1/2} \quad 1/4 \\ b & \text{"} \quad \cancel{1/4} \quad 1/8 \\ c & \text{"} \quad \cancel{1/8} \quad 1/2 \\ d & \text{"} \quad \cancel{1/8} \quad 1/8 \end{cases}$$

still

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{8}\log_2\frac{1}{8}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8} = 1 + \frac{3}{4} = \frac{7}{4}.$$

Joint Entropy:

$$H(X,Y) = -\sum_x \sum_y p_{X,Y}(x,y) \log_2 p_{X,Y}(x,y) = \mathbb{E}\left[-\log_2 p_{X,Y}(X,Y)\right]$$

Ex. Let X, Y have the following joint pmf matrix:

$$\begin{array}{c|cccc|c} x \backslash y & 1 & 2 & 3 & 4 & p_X(x) \\ \hline 1 & 1/8 & 1/16 & 1/16 & 1/4 & 1/2 \\ 2 & 1/16 & 1/8 & 1/16 & 0 & 1/4 \\ 3 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ 4 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ \hline p_Y(y) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \end{array}$$

$$H(X) = \frac{7}{4}$$

$$H(Y) = 2$$

$$H(X,Y) = -\frac{1}{4}\log_2\frac{1}{4} - \frac{2}{8}\log_2\frac{1}{8} - \frac{6}{16}\log_2\frac{1}{16} - \frac{4}{32}\log_2\frac{1}{32}$$

$$= \frac{1}{2} + \frac{3}{4} + \frac{3}{2} + \frac{5}{8} = \frac{4+6+12+5}{8} = \frac{27}{8} \text{ bits}$$

Conditional Entropy

$$\begin{tikzcd} p_Y(y) \arrow[r] \arrow[d] & p_{Y|X}(y|x) \arrow[d] \\ H(Y) & H(Y|x) = -\sum_y p_{Y|X}(y|x) \log_2 p_{Y|X}(y|x) \end{tikzcd}$$

$$H(Y|X) = \sum_x p_X(x) H(Y|x) = -\sum_x p_X(x) \sum_y p_{Y|X}(y|x) \log_2 p_{Y|X}(y|x)$$

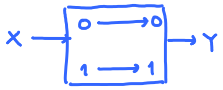
$$= -\sum_x \sum_y p_{X,Y}(x,y) \log_2 p_{Y|X}(y|x) = \mathbb{E}\left[-\log_2 p_{Y|X}(Y|X)\right]$$

$$= \mathbb{E}\left[-\log_2 \frac{p_{X,Y}(X,Y)}{p_X(X)}\right]$$

$$= \mathbb{E}\left[-\log_2 p_{X,Y}(X,Y)\right] - \mathbb{E}\left[-\log_2 p_X(X)\right]$$

$$= H(X,Y) - H(X)$$

Ex



$$\left.\begin{array}{l} H(Y|0) = 0 \\ H(Y|1) = 0 \end{array}\right\} \rightarrow H(Y|X) = \sum_x p_X(x)\, 0 = 0$$

$p_X(x)$

$$H(X|y) = -\sum_x p_{X|Y}(x|y) \log_2 p_{X|Y}(x|y)$$

$$H(X|Y) = \sum_y p_Y(y) H(X|y)$$

Ex.

$$P_{X,Y}(x,y) \quad \begin{array}{c|cccc|c} x \backslash y & 1 & 2 & 3 & 4 & P_X(x) \\ \hline 1 & 1/8 & 1/16 & 1/16 & 1/4 & 1/2 \\ 2 & 1/16 & 1/8 & 1/16 & 0 & 1/4 \\ 3 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ 4 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ \hline P_Y(y) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \end{array}$$

$$P_{Y|X}(y|x) \equiv \frac{P_{X,Y}(x,y)}{P_X(x)}$$

$$\begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 1/4 & 1/8 & 1/8 & 1/2 \\ 2 & 1/4 & 1/2 & 1/4 & 0 \\ 3 & 1/4 & 1/4 & 1/2 & 0 \\ 4 & 1/4 & 1/4 & 1/2 & 0 \end{array}$$

$H(Y|1) = \frac{7}{4}$

$H(Y|2) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{2}{4}\log_2\frac{1}{4} = \frac{3}{2}$

$H(Y|3) = \frac{3}{2}$

$H(Y|4) = \frac{3}{2}$

$$H(Y|X) = \frac{1}{2}\times\frac{7}{4} + \frac{1}{4}\times\frac{3}{2} + \frac{1}{8}\times\frac{3}{2} + \frac{1}{8}\times\frac{3}{2}$$

$$= \frac{13}{8}$$

$$P_{X,Y}(x,y) \quad \begin{array}{c|cccc|c} x \backslash y & 1 & 2 & 3 & 4 & P_X(x) \\ \hline 1 & 1/8 & 1/16 & 1/16 & 1/4 & 1/2 \\ 2 & 1/16 & 1/8 & 1/16 & 0 & 1/4 \\ 3 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ 4 & 1/32 & 1/32 & 1/16 & 0 & 1/8 \\ \hline P_Y(y) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \end{array}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 1/2 & 1/4 & 1/4 & 1 \\ 2 & 1/4 & 1/2 & 1/4 & 0 \\ 3 & 1/8 & 1/8 & 1/4 & 0 \\ 4 & 1/8 & 1/8 & 1/4 & 0 \end{array}$$

$H(X|1) = \frac{7}{4} = H(X|2)$

$H(X|3) = 2$

$H(X|4) = 0$

$$H(X|Y) = \frac{7}{4}\times\frac{1}{4} + \frac{7}{4}\times\frac{1}{4} + 2\times\frac{1}{4} + 0\times\frac{1}{4}$$

$$= \frac{11}{8}$$

$H(X) = \frac{7}{4}$

$H(Y) = 2$

$H(X,Y) = \frac{27}{8}$

$H(Y|X) = \frac{13}{8}$

$H(X|Y) = \frac{11}{8}$

→ Q: Can you find any relationship in these five numbers?

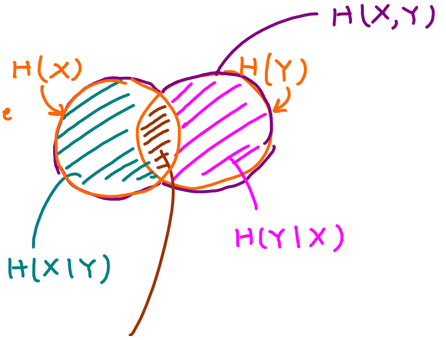
A: $H(X) + H(Y|X) = H(X,Y)$

$H(Y) + H(X|Y) = H(X,Y)$

$\Rightarrow$ $\boxed{\begin{array}{l} H(Y|X) = H(X,Y) - H(X) \\ H(X|Y) = H(X,Y) - H(Y) \end{array}}$

These provide alternative approaches for calculation of conditional entropy.

In fact, we can summarize their relationship via the Venn diagram.



$H(X|Y)$

Mutual Information $I(X;Y) = H(X) - H(X|Y)$ ←

$= \underbrace{H(Y)} - \underbrace{H(Y|X)}$ ←

$= H(X) + H(Y) - H(X,Y)$

the reduction in the randomness of one RV due to the knowledge of the other.

amount of randomness in Y

the amount of randomness still in Y when we know X

Can be regarded as the amount of information one RV contains about another RV.

Ex. Suppose $p_{X,Y} = \begin{array}{c|cc|c} x \backslash y & 0 & 1 & \\ 0 & 1/2 & 1/4 & 3/4 \\ 1 & 1/4 & 0 & 1/4 \\ \hline & 3/4 & 1/4 & \end{array}$. Find $I(X;Y)$.

$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.1226$

$H(X,Y) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{2}{4}\log_2\frac{1}{4} = 1.5$

$H(X) = -\frac{3}{4}\text{los}_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 0.8113 = H(Y)$

Ex. Suppose $p_X(x) = \begin{cases} 1/4, & x = 0 \\ 3/4, & x = 1 \\ 0, & \text{otherwise.} \end{cases}$

$p_{Y|X}(y|x)$

$x \backslash y$	0	1
0	$\frac{1}{4}$	$\frac{3}{4}$
1	$\frac{3}{4}$	$\frac{1}{4}$

$\times \frac{1}{4}$ →

$\times \frac{3}{4}$ →

⇒ $p_{X,Y}$

$p_{X,Y}(x,y) = p_X(x)\,p_{Y|X}(y|x)$

$x \backslash y$	0	1
0	$\frac{1}{16}$	$\frac{3}{16}$
1	$\frac{9}{16}$	$\frac{3}{16}$

Find $I(X;Y)$.

Method 1

$H(Y|X) = \sum_x p_X(x)\underbrace{H(Y|x)} = 0.8113 \times \overbrace{\sum_x p_X(x)}^{1} = 0.8113$

0.8113 ← Each row of $p_{Y|X}$ contains $\frac{1}{4}, \frac{3}{4}$. So, $H(Y|x) = 0.8113$ for any x. (for any row.)

$I(X;Y) = \underbrace{H(Y)}_{0.9544} - H(Y|X) = 0.1432$

To find $H(Y)$, we need $p_Y(y)$

$p_Y(y) = \sum_x p_{X,Y}(x,y) = \underbrace{\sum_x p_X(x)\,p_{Y|X}(y|x)}$

0.9544

$$p_Y(y) = \sum_x p_{X,Y}(x,y) = \underbrace{\sum_x p_X(x)\, p_{Y|X}(y|x)}$$

This is done automatically when we multiply the row vector of pmf of X and the conditional matrix $P_{Y|X}$.

$$P_X \times P_{Y|X} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{16} & \frac{6}{16} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix}$$

Method 2 First, convert the given information into the joint pmf matrix.

$$p_X(x) = \begin{cases} 1/4, & x=0 \\ 3/4, & x=1 \\ 0, & \text{otherwise.} \end{cases}$$

$H(X) = 0.8113$

$P_{Y|X}(y|x)$

$x \backslash y$	0	1
0	$\frac{1}{4}$	$\frac{3}{4}$
1	$\frac{3}{4}$	$\frac{1}{4}$

$\times \frac{1}{4}$

$\times \frac{3}{4}$

$\Rightarrow$

$P_{X,Y}$

$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y|x)$

$x \backslash y$	0	1
0	$\frac{1}{16}$	$\frac{3}{16}$
1	$\frac{9}{16}$	$\frac{3}{16}$

$\rightarrow H(X,Y) = H\left(\left[\frac{1}{16}, \frac{3}{16}, \frac{9}{16}, \frac{3}{16}\right]\right)$

$= 1.6226$

(sum along each column)

$\frac{10}{16}$ $\frac{6}{16}$ $\rightarrow H(Y) = H\left(\left[\frac{5}{8}, \frac{3}{8}\right]\right) = 0.9544$

Then, use the same calculation as in the previous example.

$$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.1432.$$

Remark:

consider X with
$$p_X(x) = \begin{cases} 1/10, & x=0, \\ 2/10, & x=1, \\ 3/10, & x=2, \\ 4/10, & x=3, \\ 0, & \text{otherwise} \end{cases}$$

$$H(X) = -\frac{1}{10}\log_2\frac{1}{10} - \frac{2}{10}\log_2\frac{2}{10} - \frac{3}{10}\log_2\frac{3}{10} - \frac{4}{10}\log_2\frac{4}{10}$$

$$= H\left(\left[\frac{1}{10} \ \frac{2}{10} \ \frac{3}{10} \ \frac{4}{10}\right]\right)$$

and Y, Z with

$P_{Y,Z}$

$y \backslash z$	0	1
0	1/10	2/10
1	3/10	4/10

$$H(Y,Z) = -\frac{1}{10}\log_2\frac{1}{10} - \frac{2}{10}\log_2\frac{2}{10} - \frac{3}{10}\log_2\frac{3}{10} - \frac{4}{10}\log_2\frac{4}{10}$$

$$= H\left(\left[\tfrac{1}{10}\ \tfrac{2}{10}\ \tfrac{3}{10}\ \tfrac{4}{10}\right]\right)$$

$$I(X;Y) = H(Y) - H(Y|X)$$

Important Properties:

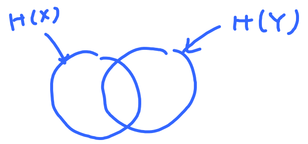
① $0 \leqslant H(X) \leqslant \log_2 |S_X|$

with equality iff X is deterministic (degenerated)

with equality iff X is uniform

② $H(X,Y) \leqslant H(X) + H(Y)$

with equality iff $I(X;Y) = 0$ iff $X \perp\!\!\!\perp Y$



③ $H(X|Y) \neq H(Y|X)$

④ $I(X;Y) \geqslant 0$

$H(Y|X) \leqslant H(Y)$

$H(X|Y) \leqslant H(X)$

with equality iff $X \perp\!\!\!\perp Y$

⑤ $I(X;Y) = I(Y;X)$

⑥ $I(X;Y) \leqslant \min\{H(X), H(Y)\} \leqslant \min\{\log_2|S_X|, \log_2|S_Y|\}$

⑦ $I(X;X) = H(X) - \underbrace{H(X|X)}_{0} = H(X)$

Quiz 4 Solution

Tuesday, September 17, 2013
3:44 PM

① Suppose $P_{X,Y} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}$. Find $I(X;Y)$. ≈ 0.02

(4.5 pt)

$$H(X,Y) = H\left(\left[\tfrac{1}{5}\ \tfrac{2}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\right]\right) = 1.9219$$

$$P_{X,Y} = \begin{array}{l} x\backslash y \\ \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix} \end{array} \begin{matrix} \rightarrow 3/5 \\ \rightarrow 2/5 \end{matrix} \Bigg\} \Rightarrow H(X) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710$$

$$\begin{matrix} \downarrow & \downarrow \\ \frac{2}{5} & \frac{3}{5} \end{matrix} \Rightarrow H(Y) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.019973 \approx 0.0200.$$

② Suppose $p_X(x) = \begin{cases} 1/5, & x = 0 \\ 4/5, & x = 1 \\ 0, & \text{otherwise} \end{cases}$ and $P_{Y|X}$ $\begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array}$.

(4.5 pt)

Find $I(X;Y)$. ≈ 0.0215

$P_{Y|X}$

$$\begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array} \begin{matrix} \rightarrow H(Y|0) = H\left(\left[\tfrac{1}{5}\ \tfrac{4}{5}\right]\right) \approx 0.7219 \\ \rightarrow H(Y|1) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) \approx 0.9710 \end{matrix} \Bigg\} \Rightarrow H(Y|X) = \frac{1}{5} \times H(Y|0) + \frac{4}{5} \times H(Y|1)$$

$$\approx 0.9211$$

$$\overset{p_X}{\left[1/5 \quad 4/5\right]} \overset{P_{Y|X}}{\begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{bmatrix}} = \overset{p_Y}{\left[\frac{9}{25} \quad \frac{16}{25}\right]} \Rightarrow H(Y) = H\left(\left[\tfrac{9}{25}\ \tfrac{16}{25}\right]\right) \approx 0.9427$$

$$I(X;Y) = H(Y) - H(Y|X) \approx 0.0215$$

Alternatively,

$P_{Y|X}$

$$\begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array} \begin{matrix} \xrightarrow{\times \frac{1}{5}} \\ \xrightarrow[\times 4]{} \end{matrix} \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & \frac{1}{25} & \frac{4}{25} \\ 1 & \frac{8}{25} & \frac{12}{25} \end{array} \qquad H(X,Y) = H\left(\left[\tfrac{1}{25}\ \tfrac{4}{25}\ \tfrac{8}{25}\ \tfrac{12}{25}\right]\right)$$

$$\approx 1.6431$$

$$\begin{array}{c|cc} 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array} \xrightarrow[\times \frac{4}{5}]{} \begin{array}{c|cc} 0 & 25 & 25 \\ 1 & \frac{8}{25} & \frac{12}{25} \end{array}$$

≈ 1.6431

$$\downarrow \quad \frac{9}{25} \quad \frac{16}{25} \Rightarrow H(Y) = H\left(\left[\tfrac{9}{25} \ \tfrac{16}{25}\right]\right) \approx 0.9427$$

$H(X) = H\left(\left[\tfrac{1}{5} \ \tfrac{4}{5}\right]\right)$
≈ 0.7219

$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.0215$

*③ Suppose $P_{Y|X}$: $\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}$ Find $I(X;Y)$.
(1 pt)

Remark: Normally, to calculate $I(X;Y)$ you will need both p_X and $P_{Y|X}$.
Here, there must be something special about $P_{Y|X}$ that allows you to get $I(X;Y)$ without p_X.

Intuition: The rows of $P_{Y|X}$ are all the same. This implies that knowing the value of x does not change the pmf of Y.

Since X does not give any information about Y, we expect $I(X;Y) = 0$.

Direct calculation:

$H(Y|x) = H\left(\left[\tfrac{1}{5} \ \tfrac{4}{5}\right]\right) \approx 0.7219$ for any x.

So, $H(Y|X) = \sum_x p_X(x) H(Y|x) \approx 0.7219 \underbrace{\sum_x p_X(x)}_{1} \approx 0.7219$.

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need p_Y.

Let's try $p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\overset{p_X}{\begin{bmatrix} 1-p & p \end{bmatrix}} \overset{P_{Y|X}}{\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}} = \overset{p_Y}{\begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix}} \Rightarrow H(Y) = H\left(\left[\tfrac{1}{5} \ \tfrac{4}{5}\right]\right) = H(Y|X)$$

regardless of the value of p

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.

10 Information Channel Capacity

Thursday, September 19, 2013
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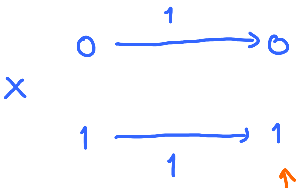
Capacity

$$C = \max_{p_X(x)} I(X;Y) = \max_{\underline{p}} I(\underline{p}, Q)$$

Ex. Noiseless Binary Channel



The binary input is reproduced exactly at the output.

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I(X;Y) = H(X) - \underbrace{H(X|Y)}_{0}$$

Given the value of Y, one can determine the value of X that produced it with 100% confidence.

$$\le \overbrace{\log_2 2}^{=1}$$ with equality iff X is uniform

$$H(X) \le \log_2 |\underbrace{S_X}|$$

there are two possible values for X: 0 and 1. So, $S_X = \{0, 1\}$ and $|S_X| = 2$.

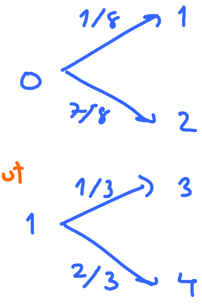
$C = 1$ bit (happens when $\underline{p} = [\frac{1}{2}\ \frac{1}{2}]$)

(bit/channel use or bit/transmission)

Interpretation: One can transmit, on average, at most one bit for each use of the channel.

Ex. Noisy channel with Non-overlapping outputs.

This channel may appear to be noisy, but really is not.



$$Q = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$I(X;Y) = H(X) - \underbrace{H(X|Y)}_{0}$$

Again, given Y, the value of X can be determined with 100% confidence.

$$\le \log_2 2$$ with equality when $\underline{p} = [\frac{1}{2}\ \frac{1}{2}]$.

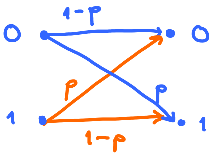
Even though the output of the channel is a random consequence of the input, the input can be determined from the output, and hence of the input, the input can be determined from the output, and hence every transmitted bit can be recovered without error.

$\leq \log_2 2$ with equality when $\underline{p} = [\frac{1}{2} \ \frac{1}{2}]$.

$C = 1$ (happens when $\underline{p} = [\frac{1}{2}, \frac{1}{2}]$)

unit: bit/channel use or bit/transmission.

Ex. BSC : Binary Symmetric Channel



$$Q = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

$$I(X;Y) = H(Y) - \underbrace{H(Y|X)}_{} = H(Y) - H(p) \overset{\text{with "=" iff } Y \text{ is uniform}}{\leq} 1 - H(p)$$

$$= \sum_x p_X(x) H(Y|x) = \sum_x p_X(x) H(p)$$

$$= H(p)$$

$$H(X) = \sum_x - p_X(x) \log_2 p_X(x)$$

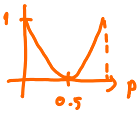
$$H(p) = -(1-p)\log_2(1-p) - p\log_2 p$$

$$H(\underline{p}) = \sum_i - p_i \log_2 p_i$$

If uniform X is used, $\underline{p} = [\frac{1}{2} \ \frac{1}{2}]$

$$\underline{q} = \underline{p}Q = [\tfrac{1}{2} \ \tfrac{1}{2}] \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} = [\tfrac{1}{2} \ \tfrac{1}{2}]$$

$C = 1 - H(p)$ bit/transmission (or bit/channel use)



(happens when $\underline{p} = [\frac{1}{2} \ \frac{1}{2}]$.)

The BSC is a special case of a class of channels called symmetric channel.

Ex. Symmetric Channel : ① All the rows of Q are permutation of each other and

② so are the columns.

Ex. $Q = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$ ✓ $\qquad Q = \xcancel{\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}}$

$C = \log_2 3 - H([0.3 \ 0.2 \ 0.5])$

$Q = \begin{bmatrix} 1/3 & 1/6 & 1/2 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$ Not symmetric

Weakly symmetric

$$I(X;Y) = H(Y) - \underbrace{H(Y|X)}_{}$$

$$= \sum_x p_X(x) \underbrace{H(Y|x)}_{}$$

$$= H([0.3 \ 0.2 \ 0.5])$$

$$= H(\underline{r})$$

$$= H(Y) - H(\underline{r}) \le \left(\log_2 |S_Y|\right) - H(\underline{r})$$

"="

To maximize H(Y), we want Y to be uniform.

Try uniform X.

(i) $$\begin{bmatrix} \frac{1}{|S_X|} & \frac{1}{|S_X|} & \cdots & \frac{1}{|S_X|} \end{bmatrix} \begin{bmatrix} \boxed{\quad \underline{r} \quad} \\ \vdots \\ \boxed{\qquad} \end{bmatrix}$$

$$\underline{p} \qquad \times \quad Q \qquad\qquad = \underline{q}$$

(ii) $$p_Y(y) = \sum_x p_{X,Y}(x,y) = \sum_x p_X(x) \overbrace{p_{Y|X}}^{Q}(y|x)$$

$$= \frac{1}{|S_X|} \underbrace{\sum_x Q(y|x)}$$

if the channel is "symmetric" or "weakly symmetric", then this sum will be the same for all y.

$C = \log_2 |S_Y| - H(\underline{r})$ for any "weakly symmetric" channel

↳ ① every rows of Q are permutation of each other and

② all column sums are equal.

"Easy" multiple-choice question

Suppose $Q = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.05 & 0.9 & 0.05 \\ 0.025 & 0.025 & 0.95 \end{bmatrix}$. Find C.

[✓ (a)] 1.0944

(b) 1.5944

(c) 2.0944

(d) 2.5944

} ← impossible

With $|S_X| = |S_Y| = 3$, we know that

$$C \le \log_2 3 \approx 1.5850$$

Quiz 5 Solution

Tuesday, September 24, 2013
3:50 PM

Evaluate the capacity values for the channels whose transition probabilities are given by each of the following matrices Q.

(a) $Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 2/3 & 1/6 & 1/6 \end{bmatrix}$

(b) $Q = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$

(d) $Q = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$